# Analysis III – Complex Analysis
# 3. Exercise Sheet

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## Groupwork

**Exercise G1** (The standard estimation)

Let $f : \mathbb{R}^n \supseteq \Omega \to \mathbb{R}^n$ be a continuous vector field and $\gamma : [0,1] \to \Omega$ a piecewise continuously differentiable path. Show the following estimation:

$$\left| \int_\gamma f\, ds \right| \leq \max\left\{ \left\| f(\gamma(t)) \right\|_2 : 0 \leq t \leq 1 \right\} \cdot L(\gamma),$$

where $L(\gamma)$ denotes the legth of $\gamma$.

**Exercise G2** (Winding around the origin)

Consider the vector field

$$f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2, \quad f\begin{pmatrix} x \\ y \end{pmatrix} := \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix}$$

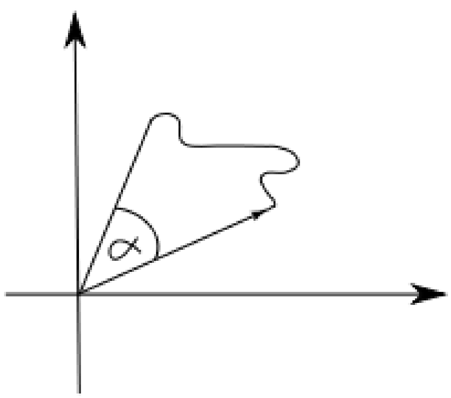
and the two star shaped domains $G_1 := \mathbb{R}^2 \setminus \{(x,0) : x \leq 0\}$ and $G_2 := \mathbb{R}^2 \setminus \{(x,0) : x \geq 0\}$.

(a) Show that $f$ has a potential on $G_1$ and a potential on $G_2$ and determine them.

**Hint:** Use the polar decomposition: $\gamma(t) = r(t) \cdot \begin{pmatrix} \cos(\varphi(t)) \\ \sin(\varphi(t)) \end{pmatrix}$.

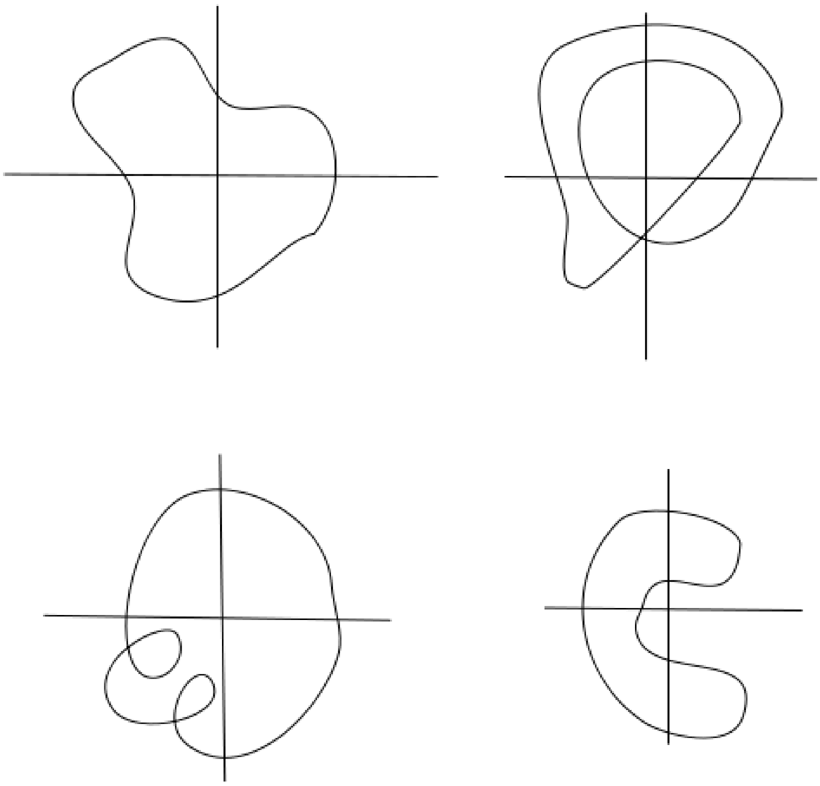
(b) Is there a global potential for $f$ on $\mathbb{R}^2 \setminus \{0\}$?

(c) Consider paths of the following form:



Show for the (improper) path integral that $\int_\gamma f\,ds = \alpha$, where $\alpha \in [0, 2\pi[$ is the included angle of the path.

(d) Determine the path integral for the following counterclockwise parametrised curves:



The number $\frac{1}{2\pi}\int_\gamma f\,ds$ is called the *winding number* of $\gamma$ in 0 for a loop $\gamma$ . Why?

**Exercise G3** (Path integrals and potentials)

In the lectures we will see that a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ has a potential if and only if its Jacobian $J_f(x)$ is symmetric for all $x \in \mathbb{R}^n$.

Decide whether the following vector fields have a potential. Determine the potential if it exists.

$$\begin{aligned}
f : \mathbb{R}^2 \to \mathbb{R}^2, \qquad f(x,y) &:= (2xy^3 + \cos(x),\ 3x^2y^2 + \cos(y))^T,\\
g : \mathbb{R}^3 \to \mathbb{R}^3, \qquad g(x,y,z) &:= (1 + y(1+x),\ x(1+z),\ xy)^T,\\
h : \mathbb{R}^2 \to \mathbb{R}^2, \qquad h(x,y) &:= (y \cdot e^{xy},\ x \cdot e^{xy} + 1)^T.
\end{aligned}$$

## Homework

**Exercise H1** (Equivalence of paths) (1 point)

(a) Show that the equivalence of paths is an equivalence relation.

(b) Let $f : \mathbb{R}^n \supseteq \Omega \to \mathbb{R}^n$ be a continuous vector field. Prove that for equivalent paths $\gamma_1 : [a,b] \to \Omega$ and $\gamma_2 : [c,d] \to \Omega$ one has $\int_{\gamma_1} f\, ds = \int_{\gamma_2} f\, ds$.

**Exercise H2** (Connectedness and pathwise connectedness) (1 point)

(a) Let $\Omega \subseteq \mathbb{R}^n$ be open and fix a point $x \in \Omega$. Consider the set $G_x := \{y \in \Omega :$ there is a continuous curve $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = y\}$. Show that $G_x$ is open and closed in $\Omega$, i. e. open and closed in the metric space $(\Omega, \|\cdot\|_2)$.

(b) Let $\Omega \subseteq \mathbb{R}^n$ be open. Prove that $\Omega$ is connected if and only if $\Omega$ is pathwise connected.

(c) Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Show that for every $x, y \in \Omega$ there is a piecewise linear path $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = y$.

**Exercise H3** (Potentials) (1 point)

(a) Let $\Omega \subseteq \mathbb{R}^n$ be a domain and $f : \Omega \to \mathbb{R}^n$ be a continuous vector field. Assume $F_1 : \Omega \to \mathbb{R}$ and $F_2 : \Omega \to \mathbb{R}$ are potentials for $f$. Show that $F_1 - F_2$ is a constant function.

(b) Let $\Omega$ be an arbitrary nonempty open subset of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^n$ be a continuous vector field. Assume $F_1 : \Omega \to \mathbb{R}$ and $F_2 : \Omega \to \mathbb{R}$ are potentials for $f$. What can you say about the difference function $F_1 - F_2$?