Analysis III – Complex Analysis
2. Exercise Sheet

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Groupwork

Exercise G1 (Cauchy-Riemann differential equations I)

Consider the function $f(z) := e^z$. Use the Cauchy-Riemann differential equations to prove that f is differentiable on the whole complex plane.

Exercise G2 (Cauchy-Riemann differential equations II)

Consider the function $f(x + y \cdot i) := x^3 \cdot y^2 + x^2 \cdot y^3 \cdot i$ defined on the whole complex plane. Determine the subset $\Omega \subseteq \mathbb{C}$ on which f has a complex derivative. Is there an inner point $z_0 \in \Omega$?

Exercise G3 (Path integrals)

Consider the vector field

$$\mathbb{R}^2 \ni (x, y) \to F(x, y) := \frac{1}{(x^2 + y^2 + 1)^2} \begin{pmatrix} -x^2 + y^2 + 1 \\ -2xy \end{pmatrix} \in \mathbb{R}^2.$$

Determine $\int_{\gamma_1} F ds$ and $\int_{\gamma_2} F ds$ for the paths $\gamma_1 : [-1, 1] \to \mathbb{R}^2$ and $\gamma_2 : [0, \pi] \to \mathbb{R}^2$ given by

$$\gamma_1(t) := \begin{pmatrix} -t \\ 0 \end{pmatrix} \quad \text{and} \quad \gamma_2(t) := \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}.$$

Exercise G4 (Elementary properties of the path integral)

Let $F,\ G : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable vector fields. Further let $\gamma, \gamma_1 : [a, b] \to \mathbb{R}^n$ and $\gamma_2 : [b, c] \to \mathbb{R}^n$ be continuously differentiable paths. Show that the path integral has the following properties:

(a) $\int_\gamma \lambda F + \mu G ds = \lambda \int_\gamma F ds + \mu \int_\gamma G ds$.

(b) $\int_{\gamma_1 + \gamma_2} F ds = \int_{\gamma_1} F ds + \int_{\gamma_2} F ds$.

(c) If $\varphi : [\alpha, \beta] \to [a, b]$ is a diffeomorphism with $\varphi'(t) > 0$ then $\int_\gamma F ds = \int_{\gamma \circ \varphi} F ds$.

Interprete part (c) in the special case of a “vector field” $F : \mathbb{R} \supseteq [a, b] \to \mathbb{R}$ and the path $\gamma : [a, b] \to \mathbb{R}$, $\gamma(t) = t$.

Exercise G5 (Rotation of a vector field and a two dimensional version of Stoke's theorem)

Let $\Omega \subseteq \mathbb{R}^2$ be an open subset and $f : \mathbb{R}^2 \supseteq \Omega \to \mathbb{R}^2$ be a continuously differentiable vector field. Further let $v \in \Omega$ be an arbitrary point and $\varepsilon > 0$. Assume that the closed square with side length ε and center v is contained in Ω and let γ be the canonical parametrisation of the boundary of this square, i. e. it is counterclockwisely orientated.

(a) Prove that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_\gamma f\, ds = \mathrm{rot}(f)(v),$$

where $\mathrm{rot}(f)(x,y) := \frac{\partial f_2}{\partial x}(x,y) - \frac{\partial f_1}{\partial y}(x,y)$ defines the rotation of f.

(b) Prove Stoke's theorem in the two dimensional case:
Let $f : \mathbb{R}^2 \supseteq \Omega \to \mathbb{R}^2$ be a continuously differentiable vector field and $R := [a,b] \times [c,d]$ be a rectangle with $R \subseteq \Omega$. If γ is the canonical parametrisation of the boundary of R then the following equation holds:

$$\int_\gamma f\, ds = \int_c^d \int_a^b \mathrm{rot}(f)(x,y)\, dx\, dy.$$

Hint: Use Fubini's theorem.

Homework

Exercise H1 (Connectedness and path-connectedness) (1 point)

Let (X,d) a metric space. The space X is called *connected*, if the only subsets of X which are both open and closed are X and the empty set.

(a) Prove that the following conditions are equivalent:
   (i) The space X is connected.
   (ii) If $X = A \cup B$ for open sets A and B with $A \cap B = \emptyset$, then $A = \emptyset$ or $B = \emptyset$.
   (iii) If $X = A \cup B$ for closed sets A and B with $A \cap B = \emptyset$, then $A = \emptyset$ or $B = \emptyset$.
   (iv) Every continuous function $f : X \to \{0, 1\}$ is constant.

(b) Is there a metric on $\mathbb{R}$ such that $(\mathbb{R}, d)$ is disconnected, i. e. not connected? Prove your claim.

(c) Show that every path connected metric space is connected.

(d) Let

$$\Gamma := \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right)^T : 0 < x \leq 1 \right\} \subseteq \mathbb{R}^2.$$

Define $X := \overline{\Gamma}$ where the closure is taken in the natural metric. Then (X, d) is a metric space with $d(x, y) := \left\| x - y \right\|_2$. Sketch the set X and show that X is connected but not path connected.

Exercise H2 (Curves, path length and rectifiability I) (1 point)

We first introduce some notation. A *partition* Z of $[0, 1]$ is given by a finite ordered subset $Z = \{t_0, ..., t_n\}$ with $0 = t_0 < t_1 < t_2 < ... < t_n = 1$. For simplicity we write $Z = \{t_0, ..., t_n\}$.
Let $\gamma : [0, 1] \to \mathbb{R}^n$ be a continuously differentiable path and Z a Partition of $[0, 1]$. We define piecewise a new path $\gamma_Z : [0, 1] \to \mathbb{R}^n$: For $t \in [t_n, t_{n+1}]$ we set

$$\gamma_Z(t) := \frac{t_{n+1} - t}{t_{n+1} - t_n} \cdot \gamma(t_n) + \frac{t - t_n}{t_{n+1} - t_n} \cdot \gamma(t_{n+1}).$$

Then γ_Z approximates γ by a polygon.
To understand this we consider an example: Let $\gamma : [0, 1] \to \mathbb{R}^2$ defined by

$$\gamma(t) := \begin{pmatrix} \cos(\pi \cdot t) \\ \sin(\pi \cdot t) \end{pmatrix}.$$

Let Z_n be the partitions $\left\{0, \frac{1}{n}, \frac{2}{n}, ..., 1\right\}$.

(a) Visualise the path γ and the paths γ_{Z_2} and γ_{Z_3}.

(b) Determine the length $L(\gamma)$ and $L(\gamma_{Z_n})$ for each $n \in \mathbb{N} \setminus \{0\}$.

(c) Show that $L(\gamma) = \lim_{n \to \infty} L(\gamma_{Z_n})$.

Remark: Let $\gamma : [0, 1] \to \mathbb{R}^n$ a path which is continuously differentiable except in finitely many points, then the length of γ is defined by

$$L(\gamma) := \int_0^1 \left\| \gamma'(t) \right\| dt.$$

Exercise H3 (Curves, path length and rectifiability II) (1 point)

Let $\gamma : [0,1] \to \mathbb{R}^n$ be a path. We call γ rectifiable, if the following supremum exists:

$$l(\gamma) = \sup\{L(\gamma_Z) : \; Z \text{ is a partition of } [0,1]\}.$$

Let Z be a partition of $[0,1]$. We call a partition Z' of $[0,1]$ a refinement of Z, if $Z \subseteq Z'$ and write $Z \leq Z'$. The *mesh* $|Z|$ of a partition $Z = \{0 = t_0, t_1, ..., t_n = 1\}$ is defined by

$$|Z| := \max\{t_{k+1} - t_k : \; 0 \leq k \leq n-1\}.$$

(a) Show that for each refinement $Z \leq Z'$ one has $L(\gamma_Z) \leq L(\gamma_{Z'})$.

(b) Show that every continuously differentiable path is rectifiable with $l(\gamma) = L(\gamma)$.