Analysis III – Complex Analysis
1. Exercise Sheet

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Groupwork

Exercise G1 (Power series of real functions)

We consider the following functions which are defined on the whole real axis:

$$f_1(x) := \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0, \end{cases} \qquad f_2(x) := \frac{x^2}{1+x^2}, \qquad f_3(x) := 1 - e^{-\frac{x^2}{2}}.$$

Sketch the graphs of these functions and expand them in $x_0 = 0$ into a Taylor series. Determine for each Taylor series the greatest open subset $U \subset \mathbb{R}$ such that the series represents the function.

Exercise G2 (Complex functions and real vector fields)

We already know that $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$ as a real vector space with the canonical $\mathbb{R}$-Basis $\{1, i\}$. In this way we identify the complex numer $z = a + bi$ with the vector $\begin{pmatrix} a \\ b \end{pmatrix}$. For this exercise we call this identification the *canonical identification of* $\mathbb{C}$ *with* $\mathbb{R}^2$.
Now we consider the complex polynomial $f : \mathbb{C} \to \mathbb{C}$ with $f(z) := z^2 + 1$.

(a) Show that f is complex differentiable in the following sense: For each complex number $z \in \mathbb{C}$ the limit

$$f'(z) := \lim_{\omega \to 0} \frac{f(z+\omega) - f(z)}{\omega}$$

exists. Calculate $f'(z)$ explicitely.

(b) We define the real vector field

$$F(x,y) := \begin{pmatrix} \mathrm{Re}(f(x + y \cdot i)) \\ \mathrm{Im}(f(x + y \cdot i)) \end{pmatrix}.$$

Show that this vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ is everywhere differentiable and calculate the Jacobian.

(c) Is there some remarkable relation of the Jacobian $J_F(x,y)$ and the value of $f'(x + yi)$?
Hint: Any complex linear funcion $T : \mathbb{C} \to \mathbb{C}$ is of course a real linear function.

Exercise G3 (Fields, matrices and complex numbers)

Let $\mathbb{K}$ be a field and let $\lambda \in \mathbb{K}$ be a number which has no square root in $\mathbb{K}$, i. e. there is no element $\mu \in \mathbb{K}$ with $\mu^2 = \lambda$.
Let $M_2(\mathbb{K})$ be the set of all 2×2 matrices with entries in $\mathbb{K}$. In this exercise we consider the subset

$$\mathbb{L} := \left\{ \begin{pmatrix} a & \lambda \cdot b \\ b & a \end{pmatrix}, \quad a, b \in \mathbb{K} \right\} \subseteq M_2(\mathbb{K}).$$

(a) Show that $\mathbb{L}$ is a field with the usual matrix addition and matrix multiplication. Assure yourself that

$$\mathbb{K} \ni x \to \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} =: x \cdot \mathbb{1} \in \mathbb{L}$$

defines an injective field homomorphism.
Hint: You may use your knowledge of matrices over fields to avoid proving every axiom for a field.

(b) in which way is $l := \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}$ special?

(c) What can you say about the eigenvalues of $a \cdot \mathbb{1} + b \cdot l$?

(d) Find a subset of $M_2(\mathbb{R})$ which is isomorphic to $\mathbb{C}$.

(e) Is there a field with 9 elements?

Exercise G4 (Visualisation of complex functions)

Consider the complex polynomial $f : \mathbb{C} \to \mathbb{C}$ with $f(z) = z^2$ and the following subset M of $\mathbb{C}$:

$$M := \{z \in \mathbb{C} : \ 0 \leq \operatorname{Re}(z) \leq 1, \ 0 \leq \operatorname{Im}(z) \leq 1\}.$$

(a) Is M open, closed, bounded, compact, convex?

(b) Calculate the image $f(M)$ and visualize the action of f by laying a grid into M, paramterizing grid lines by paths and calculating the image under f of these paths. Draw them into a draft and look on the angles of intersecting image paths. Looks something particular?

(c) What is the image of the half disk $\{z \in \mathbb{C} : \operatorname{Im}(z) \geq 0 \text{ and } |z| < 1\}$?

Homework

Exercise H1 (Curves and path length) (1 point)

Let $\gamma : [a,b] \to \mathbb{R}^n$ be a regular path which parameterises a curve $\Gamma \subseteq \mathbb{R}^n$. The arc length $s : [a,b] \to \mathbb{R}$ of γ is defined as follows:

$$s(t) := \int_a^t \left\|\gamma'(x)\right\| dx.$$

(a) Calculate $s(t)$ for the path $\gamma : [1,2] \to \mathbb{R}^3$ with $\gamma(t) := \begin{pmatrix} 2 \cdot t \\ t^2 \\ \ln(t) \end{pmatrix}$.

(b) Why do we assume the path beeing regular instead of continuously differentiable?

(c) Show that $s : [a,b] \to [0, l(\gamma)]$ is a diffeomorphism for a regular path. Use this for writing down a parameterisation $\phi : [0, l(\gamma)] \to \Gamma$ (The parameterisation by the arc length).

(d) Consider the curve $\Gamma := \{(x,y) \in \mathbb{R}^2 : y^3 - x^2 = 0\} \cap [-1,1] \times [-1,1]$. Is it possible to parameterise this curve continuously differentiable? Is it possible to parameterise this curve regularly? Prove your claim.

Exercise H2 (A very important vector field) (1 point)

Consider the function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ defined by $f(z) := \frac{1}{z}$.

(a) Calculate the real vector field $F : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ which describes after canconical representation of $\mathbb{R}^2$ and $\mathbb{C}$ the function f.

(b) Determine all points $(x,y) \in \mathbb{R}^2 \setminus \{0\}$ in which F is differentiable. For which points $(x,y) \in \mathbb{R}^2 \setminus \{0\}$ is the Jacobian $J_F(x,y)$ the action of a complex linear map?

Exercise H3 (Path connectedness) (1 point)

Let (X,d) be a metric space. We call a metric space *path connected* if for any two poins $x, y \in X$ there is a continuous path $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Show the following statements:

(a) Let (X,d) and $(Y,\tilde{d})$ be metric spaces and $f : X \to Y$ a surjective continuous map. Then Y is path connected if X is path connected.

(b) The set of all orthorgonal 2×2 matrices over $\mathbb{R}$ called $O_2(\mathbb{R})$ is not path connected. For this you can choose any norm on $M_2(\mathbb{R})$ to get a metric on $O_2(\mathbb{R})$: The result is independend of the chosen norm.
Hint: You can use that the coordinate evaluation maps $A \to A_{i,j}$ are continuous. By (a) it must be possible to find a path disconnected metric space (Y,d) and a surjective continuous map $f : O_2(\mathbb{R}) \to Y$.

(c) Let (X,d) and $(Y,\tilde{d})$ be metric spaces and let $\varphi : X \to Y$ be a homeomorphism. Then X is path connected iff Y is path connected.
Remark: 'Iff' means if and only if. It's a common and often used abbreviation in mathematical literature.

(d) There is no homeomorphism $f : \mathbb{R} \to \mathbb{C}$ if $\mathbb{R}$ and $\mathbb{C}$ carry the natural metric induced by the absolute value $|\cdot|$.

(e) There is no isomorphism of fields $\varphi : \mathbb{R} \to \mathbb{C}$.

(f**) There is a bijection $\Phi : \mathbb{R} \to \mathbb{C}$.

Remark: In the last steps we see an interesting fact: The real numbers and the complex numbers are different fields, different metric spaces but as sets they are equal in some sense.